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RDT & E Project No. 1A013001A039

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 1220

FEGrubbs/bj
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October 1963

APPROXIMATE CIRCULAR AND NON-CIRCULAR OFFSET PROBABILITIES OF HITTING

ABSTRACT

For equal or unequal delivery errors and an offset point of aim, the chance that the burst point of a warhead occurs within a given distance of a selected point of the target is approximated by reference to the Non-central Chi-square distribution. Offset circular and non-circular probabilities of hitting for the two and three dimensional cases may thus be approximated with a single, straight-forward and rather simple technique by the use of a central Chi-square distribution with fractional number of degrees of freedom or a transformation to approximate Normality. Computations of probabilities of hitting are illustrated by examples. The approximations recommended appear to be of sufficient accuracy for many weapon systems evaluation problems.

This report supersedes BRL Memorandum Report No. 1472.

INTRODUCTORY DISCUSSION

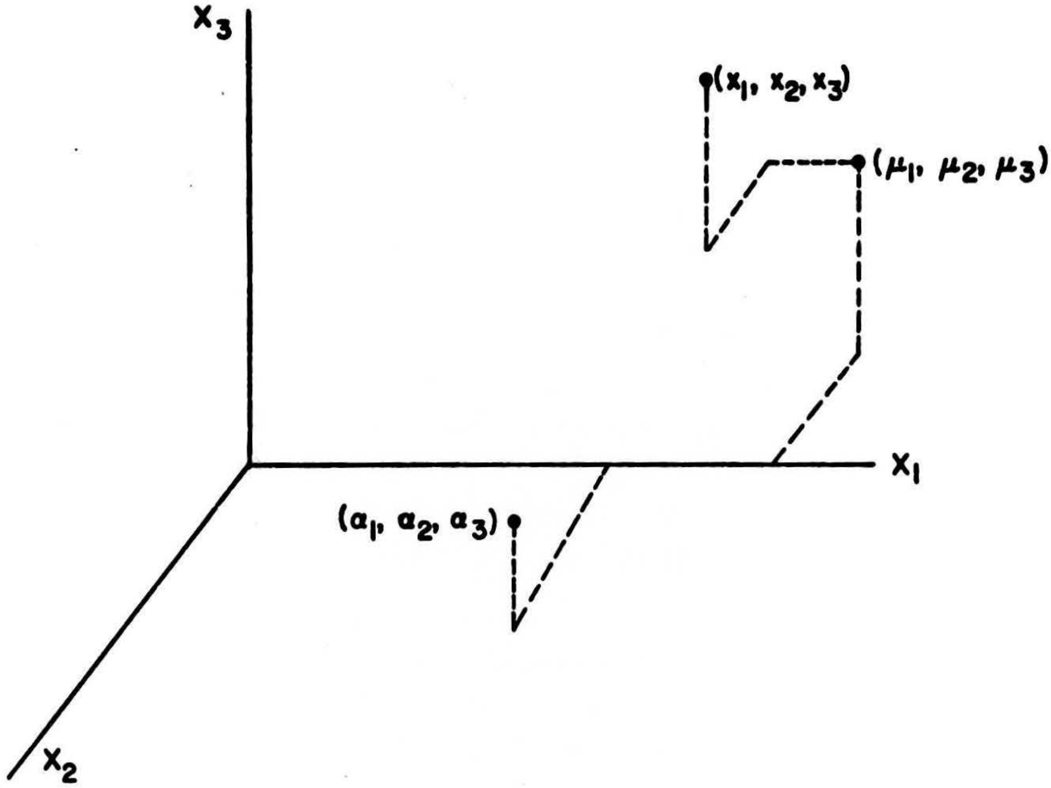
A problem of some importance to the weapon systems analyst is that of finding the probability of hitting a circular target (two-dimensional case) or a spherical target (three-dimensional case) whether the delivery errors are equal or unequal and also for point of aim or center of impact (C of I) of the rounds either coinciding with the target centroid or offset from it. Moreover, in practically all applied problems of study, the analyst does not require great accuracy in such probabilities. Certainly, two-decimal accuracy is generally good enough and in most cases probabilities of hitting in error by as much as .02, or even .03, will not be of the greatest concern. It therefore appears desirable to record a straight-forward, unique and rather simple technique for approximating probabilities of hitting for all of the various cases referred to above.

As is rather well-known, probabilities of hitting of the type discussed herein are intimately connected with the probability distributions of quadratic forms in normal variables, and a great amount of investigation has been carried out on this subject. See references [12], [9], [5], [15], [13], [6], [14], for example. In many cases for published theory on quadratic forms, however, computations of hit probabilities involve infinite series expansions or special computer programming techniques, matrix notation and very diverse theoretical ramifications which the average weapon systems analyst has little interest in or does not really need to know for many applied problems. Indeed, some simplification is desirable.

In this report we do not claim originality for any of the theory used, rather it is a matter of just how available techniques may be put together in order to obtain a single, satisfactory and useful method for approximating the probabilities desired.

THE PROBLEM AND RELATED THEORY

It is instructive to depict the problem geometrically. We will do this for the three-dimensional case and later point out how the theory may be extended to either the two-dimensional case or even the N-dimensional case. The situation is depicted in the following figure:



In the figure, the target center (or some other target point of interest) is located at $(\alpha_1, \alpha_2, \alpha_3)$, the center of impact (C of I) or aim point of the rounds at (μ_1, μ_2, μ_3) and the actual burst position of the warhead at (x_1, x_2, x_3). We assume that the burst position coordinates or the random variables x_1, x_2, x_3 are independently normally distributed with mean values and variances given by

$$E\ (x_i) = \mu_i, \quad i = 1, 2, 3 \tag{1}$$

$$\mathrm{Var}\ (x_i) = \sigma^2_{x_i} = \sigma^2_i, \quad i = 1, 2, 3 \tag{2}$$

We desire the probability that the distance $\sqrt{(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 + (x_3 - \alpha_3)^2}$ will be less than or equal to a chosen value, R. Thus, we want

$$\Pr \{(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 + (x_3 - \alpha_3)^2 \leq R^2\} \tag{3}$$

which implies we need the distribution of the quadratic form on the left-hand side of the inequality in (3). We note that this is a particular quadratic form not involving cross product terms such as $x_1 x_2$, etc.

If we let

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \quad \text{and} \quad v_i = \sigma_i^2/\sigma^2 \tag{4}$$

$$u_i = (x_i - \mu_i)/\sigma_i \quad \text{and} \quad a_i = (\mu_i - \alpha_i)/\sigma_i \tag{5}$$

then the quadratic form of (3) may be rewritten as

$$\sigma^2 \psi^2 = \sigma^2 \sum_{i=1}^{3} v_i (u_i + a_i)^2 = \sum_{i=1}^{3} (x_i - \alpha_i)^2 \tag{6}$$

Hence, we have the equivalent probability statements

$$\Pr \{\psi^2 \leq R^2/\sigma^2\} = \Pr \left\{ \sum_{i=1}^{3} (x_i - \alpha_i)^2 \leq R^2 \right\} \tag{7}$$

Now the u_i are each independently normally distributed with zero mean and unit variance. Also, the $(u_i + a_i)^2$ are each distributed as non-central Chi-square (Fisher, 1928) with one degree of freedom and non-centrality parameter a_i. Generally, if we define the non-central χ^2 as Patnaik [10], i.e.,

$$\chi'^2 = \sum_{i=1}^{\nu} (u_i + a_i)^2 \tag{8}$$

then the probability density of χ'^2 is given by [10]

$$p(\chi'^2) = \frac{e^{-\frac{\chi'^2}{2}} e^{-\frac{\lambda}{2}}}{2^{\frac{\nu}{2}}} \sum_{i=0}^{\infty} \frac{(\chi'^2)^{\frac{\nu}{2} + i - 1}}{\Gamma(\frac{\nu}{2} + i)\, 2^{2i}\, i!} \lambda^i \tag{9}$$

where $\lambda = \sum_{i=1}^{\nu} a_i^2$ is the non-central parameter and ν the number of degrees of freedom. We have the ordinary central χ^2-distribution when all the $a_i = 0$.

Hence, our generalized quadratic form (6) is merely the sum of products of constants, $v_i = \sigma_i^2/\sigma^2$, and non-central Chi-squares, i.e., a sum of weighted non-central Chi-squares. Incidentally, it is obvious that (6) generalizes to N-dimensions, although for weapon systems studies, our interest is limited to the two- and three-dimensional cases, of course. (For the two-dimensional case, we simply use i=1,2 instead of i=1,2,3.)

Patnaik [10] points out that a weighted sum of non-central Chi-squares may be approximated by fitting its first two moments to those of the ordinary central Chi-square, χ^2. Now the mean (m) and variance (v) of ψ^2 as defined in (5) are easily found, being

$$m = \sum_{i=1}^{3} v_i + \sum_{i=1}^{3} v_i a_i^2 = 1 + \sum_{i=1}^{3} (\mu_i - \alpha_i)^2/\sigma^2 \tag{10}$$

and

$$v = 2\left[\sum_{i=1}^{3} v_i^2 + 2\sum_{i=1}^{3} v_i^2 a_i^2\right] = 2\left[\sum_{i=1}^{3} \sigma_i^4/\sigma^4 + 2\sum_{i=1}^{3} \frac{\sigma_i^2}{\sigma^2}\left(\frac{\mu_i - \alpha_i}{\sigma}\right)^2\right] \tag{11}$$

Since the mean and variance of the central Chi-square (χ^2) are n and 2n degrees of freedom (d.f.), respectively, we may fit the sum of weighted non-central Chi-squares (χ'^2) to the central χ^2 by noting that the moments are proper when

$$E\left(\frac{2m\psi^2}{v}\right) = \frac{2m^2}{v} \tag{12}$$

and

$$\mathrm{Var}\left(\frac{2m\psi^2}{v}\right) = 2\left(\frac{2m^2}{v}\right) = \frac{4m^2}{v} \tag{13}$$

or $2m\psi^2/v$ is taken equal to the central χ^2 and the equivalent number of degrees of freedom is $n = 2m^2/v$.

Thus, in summary we propose to take the quantity $\chi^2 = 2m\psi^2/v$ as being approximately distributed as a central χ^2 with $n = 2m^2/v$ degrees of freedom and m and v are functions of the amount of offset and the delivery errors as given by (10) and (11). The computed value of Chi-square is taken as $\chi_o^2 = 2m\psi_o^2/v = 2mR^2/v\sigma^2$, and this last quantity is referred to a table of the probability integral of the Chi-square distribution. The desired probability is $P = P(\chi_o^2/n) = \int_o^{\chi_o^2} f(\chi^2)d\chi^2$. (See, for example, reference [1].)

Alternatively, the desired probability is given also by the following relation

$$P = I\ (\chi_o^2/\sqrt{2n},\ n/2-1) = I\ (R^2/\sigma^2\sqrt{v},\ m^2/v-1) \tag{14}$$

where I (u, p) is Karl Pearson's Incomplete Gamma Function [16]. Moreover, since there is a direct relation between the Gamma distribution, the χ^2 distribution, and the Poisson distribution, we may compute the desired probability from a table of cumulative Poisson probabilities. Letting P (c, a)* be the chance of c or more (whole) success when the expected number of successes is a, then the chance we seek is simply

$$P = P\ (c,\ a) = P\left(\frac{n}{2},\ \frac{\chi_o^2}{2}\right) \tag{15}$$

where $$P\ (c,\ a) = \sum_{r=c}^{\infty} e^{-a}\ a^r/r!$$

(In this connection, our c and a here are generally fractional values so that we merely interpolate on the tabulated whole values of c to find the desired probability.)

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* This notation conforms with that of Molina's Tables: Poisson's Exponential Binomial Limit, D. Van Nostrand Company, Inc., New York, 1942.

Finally, the Fisher transformation given in [3] or the Wilson-Hilferty transformation [17] of X^2 to approximate normal variables may be used. The Fisher transformation is

$$\sqrt{2X^2} - \sqrt{2n-1} = \text{a normal variable with zero mean and unit} \tag{16}$$

variance, or for our case, take

$$t = \sqrt{4mV^2/v} - \sqrt{4m^2/v-1} \tag{17}$$

as a normal variable: $N\ (0,\ 1)$, then

$$\Pr\left\{\sum_{i=1}^{3} (x_i - \alpha_i)^2 \leq R^2\right\} = \Pr\left\{t \leq \sqrt{4mR^2/\sigma^2 v} - \sqrt{4m^2/v-1}\right\} \tag{18}$$

or we simply refer $\sqrt{4mR^2/\sigma^2 v} - \sqrt{4m^2/v-1}$ to a table of the cumulative, standardized Normal integral.

The Wilson-Hilferty transformation [17] of X^2 to a normal variable is more accurate* and generally produces rather excellent results for cases of practical interest. The Wilson-Hilferty transformation is

$$t = \frac{\sqrt[3]{X^2/n} - (1-2/9n)}{\sqrt{2/9n}} \tag{19}$$

where t is approximately normally distributed with zero mean and unit variance: $N\ (0,\ 1)$. Thus, we refer the quantity

$$\frac{\sqrt[3]{R^2/\sigma^2 m} - (1 - v/9m^2)}{\sqrt{v/9m^2}} \tag{20}$$

to a table of the cumulative normal integral to find the desired probability.

The above discussion centered around the trivariate case. For the bivariate case, it is obvious that we delete the third terms of (3) and (6) and the last or third terms in each of the summands of (10) and (11). Otherwise, proceed as indicated.

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* Mathur [7] indicates that for three or more degrees of freedom, the Wilson-Hilferty approximation to X^2 is within .007 in probability.

In the above, we have used the first two moments (mean and variance) of a sum of weighted Non-central Chi-squares for an approximation based on the central Chi-square. Pearson [11] points out that a three-moment central Chi-square approximation for the distribution of Non-central Chi-square may be quite accurate. Returning to the quadratic form ψ^2 in Equation (6), we recall that its mean is m and variance is v, i.e. Equations (10) and (11). The third central moment of ψ^2 is easily found to be

$$\mu_3 = 8 \sum_{i=1}^{N} (v_i^3 + 3v_i^3 a_i^2) \tag{21}$$

so that Pearson's β_1 is given by

$$\beta_1 = \mu_3^2/v^3 = \left\{ 8 \sum_{i=1}^{N} (v_i^3 + 3v_i^3 a_i^2) \right\}^2 \Big/ \left\{ 2 \sum_{i=1}^{N} (v_i^2 + 2v_i^2 a_i^2) \right\}^3 \tag{22}$$

Thus, following Pearson's suggestion [11], we would propose the use of a central Chi-square approximation

$$(\chi^2_{n'} - n')/\sqrt{2n'} = (\psi^2 - m)/\sqrt{v} \tag{23}$$

where the number of degrees of freedom, n', which is generally fractional, would be obtained from

$$n' = 8/\beta_1 \tag{24}$$

The Wilson-Hilferty transformation to approximate normality could then be applied to the new $\chi^2_{n'}$, i.e. we take

$$\chi^2_{n'} = (\psi^2 - m)\sqrt{2n'/v} + n' \tag{25}$$

and refer

$$t = \left[\sqrt[3]{\chi^2_{n'}/n'} - (1-2/9n') \right] / \sqrt{2/9n'} \tag{26}$$

to a table of the Normal probability integral. (See Example 3 below for a case involving Pearson's three-moment approximation.)

DISCUSSION ON RELATED WORK

In connection with the work of this paper, Grad and Solomon [5] have investigated very thoroughly the exact and approximate distributions of quadratic forms of the type $Q = \sum_{i=1}^{k} a_i x_i^2$ in connection with probability of hitting problems. The quadratic form herein may, of course, be reduced to the form $Q = \sum_{i=1}^{k} a_i x_i^2$ by finding the appropriate eigenvalues involved. Gilliland [4] has investigated series expansions for the integral of the non-circular bivariate normal distribution over an offset circle. Also, as is no doubt well-known, Research Memorandum R-234 [8] of the Rand Corporation gives "offset circle" probabilities for the bivariate circular case. The Rand offset circle probabilities, $Q(r,\rho) = 1 - P(r,\rho)$, give for $\sigma_y = \sigma_x$ (i.e. $\sigma_2 = \sigma_1$) one minus the chance that a sample point from a bivariate circular Normal distribution will lie on or within a circle of radius r, with the aim-point offset by the distance ρ. In the notation of this paper, a very good approximation of $P(r,\rho)$ is simply $Pr(\chi^2 \leq mr^2/v\sigma_x^2)$ where $m = 1 + \rho^2/2\sigma_x^2$, $v = 1 + \rho^2/\sigma_x^2$, $\rho^2 = (\alpha_1 - \mu_1)^2 + (\alpha_2 - \mu_2)^2$ and χ^2 = Chi-square with $n = 2m^2/v$ degrees of freedom. We have checked some 100 representative values in various parts of the Rand tables. In general excellent agreement was obtained and the greatest discrepancy for the approximation based on formula (20) was .023 in probability. For the bivariate case, with $\sigma_x = \sigma_y$, Helen J. Coon of the Ballistic Research Laboratories derived the formula for estimating the offset circular hit probabilities by using the Wilson-Hilferty transformation of the non-central χ^2.

Di Donato and Jarnagin give a computer program for numerical evaluation and tables of six particular probability levels (P = .05, .20, .50, .70, .90 and .95) for the offset, non-circular bivariate case. A check indicates good agreement of our approximation with their values. (See Example 3.)

Pachares [9] gives, for the distribution of a definite quadratic form involving central normal variables, an alternating infinite series expansion which converges absolutely, and which gives an upper bound for the cumulative distribution function if one stops with an even power or a lower bound for an odd power of the series expansion. Shah and Khatri [15] generalize the result of Pachares to the case of non-central normal variables.

Ruben [13], [14] in rather elegant fashion studies in rigorous detail the probability content of regions under spherical normal distributions, the regions including half-spaces, hyperspheres, hypercones, hypercylinders, ellipsoids, simplices and polyhedral cones.

Imhof [6] discusses exact and approximate methods for computing the distribution of quadratic forms in non-central normal variables. We believe that our recommended computation based on (20) is simpler and sufficiently accurate for weapon systems evaluations.

Additional Points of Interest

a) If the target is subject to location errors, $\sigma^2_{T_{x1}}$, $\sigma^2_{T_{x2}}$, $\sigma^2_{T_{x3}}$ in addition to the ever-present round-to-round variances $\sigma^2_{x_1}$, $\sigma^2_{x_2}$, $\sigma^2_{x_3}$ about the C of I, we may put

$$\sigma_1^2 = \sigma^2_{T_{x1}} + \sigma^2_{x_1}; \ \sigma_2^2 = \sigma^2_{T_{x2}} + \sigma^2_{x_2}; \ \sigma_3^2 = \sigma^2_{T_{x3}} + \sigma^2_{x_3}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

and otherwise proceed as before in the analysis.

b) With the theory herein it is easy to determine the approximate circular or spherical probable error, even for unequal delivery errors. Equating (20) to zero and solving for R, we obtain the radial distance which includes 50 percent of the points, i.e., the circular probable error (CEP) or the spherical probable error (SEP). For example, for the bivariate circular case with no offset, we have $\sigma_1 = \sigma_2 = \sigma/\sqrt{2}$, m=v=1 and we find $R = 32/27\ \sigma_1 = 1.185\ \sigma_1$ as compared to the well-known relation $\text{CEP} = \sqrt{2 \ln 2}\ \sigma_1 = 1.178\ \sigma_1$.

c) It is noted that there is a relationship between the problem discussed here and "coverage" problems in weapon systems analysis. The so-called coverage problem involves the measure of two - and three-dimensional random sets or intersections of circles, spheres, etc. for various delivery errors. Indeed, the amount or fraction of coverage for one circle falling on another circle, for example, is determined from geometrical considerations, whereas the probability that the coverage will equal or exceed a given value may be estimated from the methods discussed herein.

d) Once probabilities of hitting are available as computed herein, lethality or vulnerability data may be easily included in order to obtain overall "kill" probabilities for a weapon.

e) Finally, it is remarked that in many weapon assessment cases it will not be of great concern whether one uses a round (or spherical) target as compared to a square target, since available vulnerability data or lethality data or other input information may lead to some lack of precision anyway.

Example 1

Using problem 1, page 764, of Gilliland's paper [4] for a bivariate example, we have

$$R = \sigma_1, \alpha_1 = \alpha_2 = 0, \mu_1 = \sigma_1/5, \mu_2 = \sigma_2/5, \sigma_2 = \sigma_1/2, \text{ and}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 = 5\sigma_1^2/4$$

From (10) and (11), for i=1, 2, we find that $m = 1.04$ and $v = 1.468$. Thus, $\chi_o^2 = 2mR^2/\sigma^2 v = 2\ (1.04)\ (.8)/1.468 = 1.134$ and $n = 2m^2/v = 1.474$. From the Biometrika Tables [1], we find our $P = P\ (\chi_o^2/n) = .570$. Equivalently, using Karl Pearson's Incomplete Gamma function, $I\ (u, p) = I\ (R^2/\sigma^2\sqrt{v}, m^2/v-1) = I\ (.660, -.263) = .573$. The Poisson approximation gives $P\ (n/2, \chi_o^2/2) = P\ (.737, .567) = .558$. These three values differ no doubt because of the double interpolation required in the various tables.

For the Wilson-Hilferty approximation (19), we refer

$$\frac{\sqrt[3]{R^2/\sigma^2 m} - (1 - v/9m^2)}{\sqrt{v/9m^2}} = .1729$$

to a table of the cumulative normal integral and we get $P = .569$, which is of sufficient accuracy. (The Fisher approximation (10) to X^2 gives $P = .544$.) In contrast to all the above probabilities Gilliland's computed value of P is .577.

Example 2

If we consider Gilliland's problem 3, page 766 of [4], the data are the same as in Example 1 except now $\sigma_1 = \sigma_2$, and we find $m = 1.04$ and $v = 1.08$. Our computed value of Chi-square is $X_o^2 = 2mR^2/v\sigma^2 = .963^-$ with an equivalent number of degrees of freedom $n = 2m^2/v = 2.003$. For these values, we find from the Biometrika Tables that $P = P\ (X_o^2/n) = .381$. Tables of the Incomplete Gamma Function give $I\ (u,\ p) = I\ (.481,\ .00148) = .381$ and interpolation in the Poisson Tables gives $P = P\ (n/2,\ X_o^2/2) = P\ (1.002,\ .481) = 381$. The Wilson-Hilferty approximation gives $P = .376$ and Gilliland's computed value is .382.

Example 3

From the Di Donato and Jarnagin tables [2], we select the following three cases.

a) For a probability $P = .50$, we have $\sigma_1 = 1$, $\sigma_2 = 3$, $\mu_1 = 1$, $\mu_2 = 4$ and $R = 4.283$. Our $m = 2.70$, $v = 7.44$, $t = -.0223$ from (20) and hence $P = .4911$, which is in error by only $-.0089$.

b) For $P = .90$, another case is $\sigma_1 = 1$, $\sigma_2 = 6$, $\mu_1 = 5$, $\mu_2 = 20$ and $R = 28.159$. Note that we have considerable offset and non-circularity. In this case, our $m = 12.486$, $v = 44.042$, $t = 1.291$ from (20) and $P = .9017$, so that the error is only $+.0017$.

c) For a very extreme case, we take $\sigma_1 = 1$, $\sigma_2 = 10$, $\mu_1 = 50$, $\mu_2 = 1$, $P = .20$ and $R = 49.696$ from the Di Donato and Jarnagin tables. We have $m = 25.762$ $v = 2.980$, $t = -.7495$ and $P = .2268$, so that the error is $+.0268$.

Had we used Pearson's three-moment approximation (25), (24) for this case, we have $\beta_1 = .3044$, $n' = 26.28$, $\chi^2_{n'} = 20.78$ and $t = -.7268$ from (26). Thus, $P = .2337$ and the error is $+.0337$, i.e. somewhat greater and at the expense of more involved computation. Nevertheless, Imhof [6] indicates that Pearson's three-moment approximation of Central χ^2 to weighted non-central χ^2 may be quite accurate generally as compared to this single case.

Example 4

Mr. H. L. Merritt of the Ballistic Research Laboratories conducted a Monte-Carlo or sampling experiment on the ORDVAC electronic computer to compare probabilities of hitting a spherical target, based on 200 trials for each probability, with that obtained by use of the Wilson-Hilferty transformation (19) of χ^2 to approximate normality. The computations were programmed by Mrs. Jean Jones employing the FORAST machine language or symbolic coding.

The "target" was a sphere of radius unity, located at the origin. Different values of σ_1^2, σ_2^2, σ_3^2, $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ and various C of I locations were employed as indicated on the following table. The sample size for each machine run to predict probability of hitting was 200 as already mentioned. The sampling results are indicated in the table below, along with the computed Wilson-Hilferty values of probability.

Table of Computed and Monte Carlo Probabilities

σ^2	C of I Locations									
	(0, 0, 0)		(1/2, 0, 0)		(1, 0, 0)		(2, 0, 0)		(3, 0, 0)	
	WH	Ma	WH	Ma	WH	Ma	WH	Ma	WH	Ma
					$\sigma_1 = \sigma_2 = \sigma_3$					
.5	.890	.885	.807	.755	.337	.353	0	0	0	0
1.0	.607	.615	.530	.505	.261	.320	.001	.020	0	0
2.0	.313	.320	.307	.315	.174	.205	.004	.035	0	0
3.0	.204	.210	.205	.195	.127	.135	.006	.040	0	.010
					$\sigma_1 = 2\sigma_2 = 2\sigma_3$					
.5	.868	.860	.739	.770	.432	.435	.019	.040	0	0
1.0	.630	.670	.554	.555	.372	.405	.055	.090	.001	0
2.0	.388	.400	.364	.400	.287	.290	.092	.125	.011	.035
3.0	.279	.250	.269	.285	.233	.235	.104	.105	.023	.050
					$\sigma_1 = \sigma_2 = 2\sigma_3$					
.5	.878	.890	.738	.735	.378	.380	.005	.015	0	0
1.0	.618	.635	.524	.520	.308	.355	.021	.040	0	0
2.0	.350	.365	.314*	.395*	.221	.265	.044	.050	.002	.010
3.0	.236	.275	.219	.270	.171	.210	.053	.070	.006	.010
					$\sigma_1 = 2\sigma_2 = 3\sigma_3$**					
.5	.740	.774	.739	.750	.447	.452	.025	.044	0	.001
1.0	.640	.662	.568	.584	.392	.408	.068	.087	.002	.009
2.0	.415	.412	.390	.366	.312	.306	.109	.122	.015	.029
3.0	.310	.275	.299	.266	.259	.223	.123	.120	.030	.046

WH = Wilson-Hilferty
Ma = Machine (ORDVAC)

* Indicates greatest discrepancy

** For this block 1000 shots for each probability were generated.

The results are generally within the expected sampling error. Since, however, the greatest discrepancy occurred for the values marked*, then 10,000 "shots" were generated or sampled on the ORDVAC for this condition and the experimental probability of hitting converged to .323 vs. the .314 for the WH approximation. The agreement is thus very good indeed.

There is some evidence that our procedure along with the Wilson-Hilferty transformation may on the average just slightly underestimate the true probabilities we seek.

Example 5

Given that the x_i or shots are normally distributed with zero means and variances σ_i^2, what is the chance that a point (round) will lie (or hit) on or within the ellipsoid

$$\sum_{i=1}^{N} \frac{x_i^2}{a_i^2} = 1 \ ?$$

Now, we want the probability

$$\Pr\left\{\sum \frac{x_i^2}{a_i^2} \leq 1\right\} = \Pr\left\{\sum \frac{\sigma_i^2}{a_i^2} u_i^2 \leq 1\right\} = \Pr\left\{\sum c_i u_i^2 \leq 1\right\}$$

where $u_i = x_i/\sigma_i$ and $c_i = \sigma_i^2/a_i^2$. The mean and variance of $\phi = \Sigma c_i u_i^2$ are easily found to be

$$m = E(\phi) = \sum c_i \qquad v = \mathrm{Var.}(\phi) = 2\sum c_i^2$$

Hence, we may take

$$\chi^2 = \frac{2m\phi}{v}$$

as being approximately distributed as Chi-square with $2m^2/v$ d.f.

Using the Wilson-Hilferty transformation, we would refer

$$t = \frac{\sqrt[3]{1/m} - (1 - v/9m^2)}{\sqrt{v/9m^2}}$$

to a table of the standardized normal integral.

Example 6

Suppose the x_i (burst positions) are independently normally distributed with unequal means (μ_i) and unequal standard deviations (σ_i) and we wish to find the probability of hitting on or within the offset ellipsoid

$$\phi = \sum (x_i - b_i)^2/a_i^2 \leq 1$$

Now putting $u_i = (x_i - \mu_i)/\sigma_i$ and $A_i = (\mu_i - b_i)/\sigma_i$ we have

$$\phi = \sum V_i \ (u_i + A_i)^2 \text{ where } V_i = \sigma_i^2/a_i^2$$

Moreover, the mean and variance of ϕ are given by

$$m = E(\phi) = \sum V_i \ (1 + A_i^2); \quad v = 2 \sum V_i^2 \ (1 + 2A_i^2)$$

and we may take $2m\phi/v$ as being approximately distributed as χ^2 with $2m^2/v$ degrees of freedom.

Other quadratic forms could be treated similarly.

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APPROXIMATE CIRCULAR AND NON-CIRCULAR OFFSET PROBABILITIES OF HITTING
F. E. Grubbs

BRL Report No. 1220 October 1963

RDT & E Project No. 1A013001A039
UNCLASSIFIED Report

Weapons evaluation - Hitting
Probability (Statistics) - Errors
Gunnery

For equal or unequal delivery errors and an offset point of aim, the chance that the burst point of a warhead occurs within a given distance of a selected point of the target is approximated by reference to the Non-central Chi-square distribution. Offset circular and non-circular probabilities of hitting for the two and three dimensional cases may thus be approximated with a single, straightforward and rather simple technique by the use of a central Chi-square distribution with fractional number of degrees of freedom or a transformation to approximate Normality. Computations of probabilities of hitting are illustrated by examples. The approximations recommended appear to be of sufficient accuracy for many weapon systems evaluation problems.

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